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STABLE PAIRS AND THEIR SPECTRAL CURVES

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Throughout this talk X will denote a compact Riemann surface of genus $g \geq 2$, and L a fixed holomorphic line bundle on X. Details of proofs may be found in [4,5].

(1) **Definition:** *A* stable pair *on* X *with respect to* L *(or* stable L-pair *for short) is a pair* (V,Φ) *where* V *is a holomorphic vector bundle on* X, *and* $\Phi: V \to V \otimes L$ *is a vector bundle homomorphism such that one has* $\mu(W) < \mu(V)$ *(where* $\mu = \frac{\text{rank}}{\text{degree}}$ *) for all* Φ-invariant *subbundles* $W \subset V$. □

(We say that $W \subset V$ is Φ-invariant if $\Phi(W) \subset L \otimes W$.)

Of course, stable pairs always exist by taking $\Phi \equiv 0$ and V to be a stable bundle. Thus the definition generalises the usual notion of stable bundle; moreover one can define isomorphism, families and the moduli problem in the obvious way. One then has the following result due to Nitin Nitsure.

(2) Theorem [3]: *There exists a coarse moduli scheme* $\mathcal{M}'$ *for stable L-pairs of fixed rank* n *and degree* d *(ie. rank and degree of the*

bundle). $\mathcal{M}'$ *is a separated Noetherian scheme of finite type over* $\mathbb{C}$. *and contains a connected component* $\mathcal{M} \subset \mathcal{M}'$ *such that* $(V,\Phi) \in \mathcal{M}$ *whenever* V *is a semistable bundle. Moreover,* $\mathcal{M}$ *is a quasi-projective variety, and is smooth and irreducible if* $\deg L \geq 2g-2$. □

In the case $\deg L \geq 2g-2$ we shall give the dimension of $\mathcal{M}$ below (remark (11)).

One may conjecture that $\mathcal{M}$ is the *only* connected component, though this is not known in general. We remark in addition that one has an identical theorem for *tracefree* stable pairs (ie. requiring $\mathrm{tr}\Phi \in H^0(X,L)$ to vanish) and we shall denote the resulting connected variety by $\mathcal{M}_0$.

We now aim to use these spaces to study the moduli space N = N(n,d) of semistable *bundles* on X (up to S-equivalence) of rank n and degree d. Recall that N is a normal projective variety, smooth on the open set N^s of stable bundles, with dimension $n^2(g-1) + 1$. We shall consider a diagram of the following form:

$$(3) \qquad \begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad h \quad} & W = \oplus_{i=1}^{n} H^0(X,L^i) \\ \Big\downarrow {\scriptstyle \text{fibre } H^0(L\otimes \mathrm{End}V) \atop \scriptstyle \text{over stable } V} & & \\ N(n,d) & & \end{array}$$

The vertical map, if well-defined, will be a rational map which forgets the homomorphism Φ when V is semistable. Clearly this will have fibre $H^0(X,L\otimes\mathrm{End}V)$ when $V \in N^s$.

The horizontal map h is defined by

$$h(V,\Phi) = (\mathrm{tr}\,\Phi,\ \mathrm{tr}\wedge^2\Phi,\ \ldots,\ \wedge^n\Phi);$$

ie. the coefficients of the characteristic polynomial of Φ. More precisely, let $Z \xrightarrow{p} X$ denote the total space of the line bundle L. Then there is a tautological section $\lambda \in H^0(Z,p^*L)$, and the characteristic polynomial is

$$\det |p^*\Phi - \lambda\otimes \mathrm{id}| \equiv \lambda^n - \lambda^{n-1}p^*a_1 + \ldots + (-)^n p^* a_n$$

where $a_i = \mathrm{tr}\Lambda^i\Phi \in H^0(X,L^i)$.

This polynomial is to be viewed as an element of $H^0(Z,p^*L^n)$, and as such has as zero-locus some curve $\tilde{X} \subset Z$, whose points represent the eigenvalues of Φ.

The above construction is due to Nigel Hitchin [1], who was interested in the case $L = K$, the canonical line bundle. We shall call the curve $\tilde{X}$ the *spectral curve* of the pair (V,Φ); we note that $\tilde{X} \in |\mathcal{O}_{\mathbb{L}}(n)|$ where $\mathbb{L}$ denotes the ruled surface $\mathbb{P}(L^{-1} \oplus \mathcal{O}_X)$ which compactifies Z. In particular, $H^0(\mathbb{L},\mathcal{O}_{\mathbb{L}}(n)) \cong H^0(X,S^n(L\oplus\mathcal{O}_X)) \cong \mathbb{C} \oplus W$ ie. W can be viewed as an affine subspace of the projective space $|\mathcal{O}_{\mathbb{L}}(n)|$. Thus we shall identify the image $h(V,\Phi)$ in (3) with the spectral curve $\tilde{X}$.

We next ask for the image $h(\mathcal{M}) \subset W$.

(4) **Definition:** $\tilde{X} = \{\lambda^{\nu} - a_1\lambda^{n-1} + \ldots + (-)^n a_n = 0\} \in W$ *will be called* regular *if it is smooth and irreducible and has only simple branching over* X, *in the sense that the discriminant* $\Delta(\lambda^n - a_1\lambda^{n-1} + \ldots) \in H^0(X,L^{n(n-1)})$ *has only simple zeros. We write* $W^{reg} \subset W$ *for the subset of regular* $\tilde{X}$. □

(5) Proposition: *Suppose (i) $|L^n|$ is a base-point-free linear series on X; and (if n>2) (ii) $H^0(X,L^{n-1}) \neq 0$. Then W^{reg} is a non-empty Zariski-open subset of W. Moreover, a regular $\tilde{X}$ has genus $\tilde{g}$* $= \frac{1}{2}n(n-1)\deg L + n(g-1) + 1$. □

Next, for each stable pair (V,Φ), the corresponding spectral curve $\tilde{X}$ comes equipped with a natural sheaf, called the *spectral sheaf*,

$$F = \ker(p^*\Phi - \lambda\otimes \mathrm{id}) \subset p^*V;$$

the sheaf of 'eigenvectors' of Φ. F is a torsion-free sheaf of rank 1 on $\tilde{X}$ and is a line bundle when $\tilde{X}$ is smooth.

One expects to be able to reconstruct (V,Φ) locally from its eigenvalues and eigenvectors, where these are linearly independent, and analytically continue over all of X. The result is the following.

(6) Proposition: *If $\tilde{X}$ is regular then* $V = (p_*F^{-1})^\vee$, *and Φ is adjoint to the homomorphism $V^\vee \to L\otimes V^\vee$ obtained by pushing down the tautological homomorphism* $F^{-1} \xrightarrow[\otimes\lambda]{} (p^*L)\otimes F^{-1}$. □

(7) *Remark:* The idea of the proof is to regard the line bundle F as giving a section of the projectivised bundle $\mathbb{P}(p^*V) \to \tilde{X}$. This section has an image $\mathfrak{X} \subset \mathbb{P}(V)$ via the diagram

$$\begin{array}{ccc} \mathbb{P}(p^*V) & \to & \mathbb{P}(V) \\ F\uparrow\downarrow & & \downarrow q \\ \tilde{X} & \xrightarrow{p} & X \end{array}$$

$\mathfrak{X}$ has a natural morphism $\pi:\mathfrak{X} \to \tilde{X}$ such that π^*F is isomorphic to the restriction of the tautological line bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ to $\mathfrak{X}$, and one checks that $p^*F^{-1} \cong q_*\pi^*F^{-1} \cong q_*\mathcal{O}_{\mathbb{P}(V)}(1) \cong V^\vee$.

The statement about Φ then follows easily. $\square$

One thus has a correspondence between L-pairs (V,Φ) and spectral pairs $(\tilde{X},F)$. The next lemma shows that this inverse construction even yields, for general F, a pair (V,Φ) for which V is a stable bundle.

(8) **Lemma:** *Suppose* $\tilde{X}$ *is regular, and for each* $i = 1, \ldots, n-1$ *let*

$$S_i = \{F \in \operatorname{Pic} \tilde{X} \mid V := (p_* F^{-1})^\vee \textit{ has a destabilising subbundle } W \subset V \textit{ of rank } i\}$$

('destabilising' means $\mu(W) \geq \mu(V)$.) *Then* S_i *is a closed subvariety of* $\operatorname{Pic} \tilde{X}$ *with codimension* $\geq (2.\binom{n}{i}-n-1)(g-1)$.

In particular V *is stable away from a subvariety of codimension* $\geq (n-1)(g-1)$. $\square$

(9) **Theorem:** *If* $\tilde{X}$ *is regular then the fibre* $h^{-1}(\tilde{X})$ *of the horizontal map in the diagram* (3) *is isomorphic to* $\operatorname{Jac}(\tilde{X})$, *and contains a closed subvariety* S *of codimension* $\geq (n-1)(g-1)$ *whose complement consists of points corresponding to stable pairs* (V,Φ) *for which* V *is a stable bundle.*

The idea of the proof is of course to construct a family of L-pairs (V,Φ) on X, parametrised by $\operatorname{Jac}(\tilde{X})$, by the prescription of proposition (6). (Note that the degree of F is fixed by the requirement that $\deg V = d$.) These pairs automatically have spectral curve $\tilde{X}$, and in particular must be stable. (This is because the curve $\mathfrak{X} \subset \mathbb{P}(V)$ of remark (7) is irreducible and generically spans $\mathbb{P}(V)$, so that V cannot have *any* Φ-invariant subbundles.) Thus we get a morphism $\operatorname{Jac}(\tilde{X}) \to \mathcal{M}'$, and by lemma (8) the image lies in the component $\mathcal{X}$, hence coincides with the fibre $h^{-1}(\tilde{X})$. $\square$

As a corollary one sees that $h : \mathcal{X} \to W$ is surjective on W^{reg}. We shall consider the closure $\bar{\mathcal{X}}^{reg} \subset \mathcal{X}$ of the open subvariety $\mathcal{X}^{reg} := h^{-1}(W^{reg})$, which is an irreducible component of $\mathcal{X}$. If L is sufficiently positive in the sense of proposition (5) then the diagram (3) can be replaced by:-

(10)
$$\begin{array}{ccc} \begin{smallmatrix}\text{irred}\\ \text{quasi}\\ \text{proj}\end{smallmatrix}\ \bar{\mathcal{X}}^{reg} & \xrightarrow[\text{morphism}]{h \text{ dominant}} & W \subset |\mathcal{O}_L(n)| \\ \Big\downarrow {\scriptstyle \text{fibre } H^0(L\otimes \mathrm{End}V) \text{ over stable } V} & & \\ N(n,d) & & \end{array}$$

The vertical map is now a well-defined rational map, and the dimension of $\bar{\mathcal{X}}^{reg}$ can be computed:

$$\dim \bar{\mathcal{X}}^{reg} = \dim W + \tilde{g}$$
$$= n^2 \deg L + 1 + \textstyle\sum_{i=1}^{n} h^i(X, L^i).$$

(11) *Remark:* Note that the positivity conditions of proposition (5) are satisfied by all L for which $\deg L \geq \frac{2g}{n}$; and in particular all L for which $\deg L \geq 2g-2$ (if $g\geq 2$, $n\geq 2$). In this case theorem (2) says $\mathcal{X} = \bar{\mathcal{X}}^{reg}$ and is smooth; thus in this case one has

$$\dim \mathcal{X} = n^2 \deg L + 1 + h^1(X,L). \quad \square$$

We shall give two examples of the above construction.

Example 1: L = K

K denotes the canonical line bundle on X. When n=2 and d is odd, this is the case considered by Hitchin in [1]. By remark (11) $\mathcal{M}$ is smooth of dimension $2n^2(g-1) + 2 = 2.\dim N(n,d)$. Moreover, the vertical fibre in (10) over $V \in N^s$ is $H^0(X, K\otimes \mathrm{End}V) \cong H^1(X,\mathrm{End}V)^* \cong T_V^*N^s$, and in this way one has an inclusion $T^*N^s \subset \mathcal{M}$ as a dense open subset.

In Hitchin's original situation $\mathcal{M}$ is a moduli space for solutions of the (SU(2)-)self-dual Yang-Mills equations on X, which are identified with stable K-pairs. Hitchin showed that the canonical symplectic structure on T^*N^s extends over $\mathcal{M}$ and that with respect to this symplectic structure the components of the map $h : \mathcal{M} \to W$ all *Poisson-commute*. But $\dim \mathcal{M} = 2.\dim W$, so by Liouville's theorem of symplectic geometry the general fibre has to be a complex torus. The spectral curve construction then identifies these tori.

Note also that $\tilde{g} = \dim N(n,d)$. In fact, one can show [4] that the vertical projection on to N(n,d) restricts to a general horizontal fibre $\mathrm{Jac}(\tilde{X})$ as a dominant rational map $F \to (p_*F^{-1})^\vee$ of finite degree $1^g.2^{3g-3}.3^{5g-5}\ldots n^{(2n-1)(g-1)}$.

In the case n=2 one can use this map to reprove a classical theorem of Nagata on ruled surfaces, as follows. For any bundle $V \to X$ of rank 2 we define its *Segre invariant* to be

$$s(V) = \deg V - 2.\max \deg W \text{ ; max taken over line subbundles } W \subset V;$$
$$= \text{minimum self-intersection of a section of } \mathbb{P}(V).$$

(12) **Theorem**[2,4]: *For all* V, $s(V) \le g$.
ie. every ruled surface has a cross-section of self-intersection $\le g$.

Idea of proof: Any counterexample is necessarily stable, so it suffices to consider the filtration of $N = N(n,d)$:-

$N_0 \subset N_1 \subset \ldots$; where $N_k = \{V \mid s(V) \le 2k\}$.

One knows that the N_k are closed subvarieties, and one wants to show that the filtration terminates at $k = \left[\frac{g}{2}\right]$. One constructs a corresponding filtration on $\mathrm{Jac}(\tilde{X}) = \tilde{J} \to N$ as follows.

Consider the diagram

$$\text{(13)} \qquad \begin{tikzcd} \tilde{J} \times \mathrm{Pic}(X) \arrow[r, "\mu"] \arrow[d, "\substack{\text{projection}\\ \text{1st factor}}"] & \mathrm{Pic}(\tilde{X}) \\ \tilde{J} & \end{tikzcd}$$

where $\mu(F,W) = \sigma^* F \otimes p^*(F.W^{-1})$; and $\sigma\colon \tilde{X} \to \tilde{X}$ is the sheet-interchange over X.

Claim: $H^0(\tilde{X},\mu(F,W)) \cong H^0(X, W^{-1} \otimes V_F)$ where $V_F = (p_* F^{-1})^{\vee}$.

If we assume for simplicity that $d = \deg V = 0$, then noting that $\deg \mu(F,W) = 2g - 2 - 2.\deg W$, one sees that $V_F \in N_k \Longleftrightarrow$ $\exists W$ such that $H^0(\tilde{X},\mu(F,W)) \neq 0$ and $\deg W \ge -k \Longleftrightarrow F \in \pi\circ\mu^{-1}(W_{d(k)})$ where $d(k) = 2g - 2 + 2k$ and $W_{d(k)} \subset \mathrm{Pic}(\tilde{X})$ is the subvariety of special line bundles of degree $\le d(k)$.

Writing $\tilde{J}_k = \pi\circ\mu^{-1}(W_{d(k)})$ therefore defines a filtration of $\tilde{J}$ which matches up with that on N.

Claim: $\tilde{J}_k \to N_k$ has finite degree for each $k > 0$, and

$$\dim N_k = \dim \tilde{J}_k = \begin{cases} d(k)+g = 3g-2+2k & \text{if } 2k \le g-1 \\ 4g-3 = \tilde{g} & \text{if } 2k \ge g-1. \end{cases}$$

ie. the dimension that one would expect; this involves checking certain transversality conditions in the diagram (13). □

(14) *Remark:* One may note that $\tilde{J}_0 \to N_0$ does *not* have finite degree. This corresponds to the fact that points of N_0 represent 'positive-

dimensional' S-equivalence classes of non-stable bundles. The fibre of $\tilde{J}_0$ over such a point then consists of line bundles F which 'sweep out' the equivalence class downstairs by the prescription $F \to V_F$.

Example 2: $L^2 = K$

We shall now take for L some fixed theta-characteristic on X. Note that any such L satisfies the conditions of proposition (5). Referring again to diagram (10) one has

$$\begin{aligned} \dim \bar{\mathcal{X}}^{reg} &= n^2(g-1) + 1 + h^1(L) + 1 \\ &= \dim N(n,d) + h^0(L) + 1. \end{aligned}$$

But the term $h^0(L)$ comes from the first component of the vector space W, ie. from the *trace*. Taking tracefree stable pairs one finds that all of the above work goes over unchanged, $\bar{\mathcal{X}}^{reg}$ being replaced by $\bar{\mathcal{X}}_0^{reg}$ in (10), where now when $L^2 = K$ one has

$$\dim \bar{\mathcal{X}}_0^{reg} = \dim N(n,d) + 1.$$

The next lemma says that in fact (if n is even) the vertical map in (10) is dominant with 1-dimensional fibres.

(15) **Lemma:** *Suppose* n *is even. Then there exists a theta-characteristic* L *and a Zariski-open subset* $\mathfrak{U} \subset N^s(n,d)$ *such that if* $V \subset \mathfrak{U}$ *then* $h^0(X, L\otimes \mathrm{End}_0 V) = 1$ *and for any* $\Phi \in H^0(X, L\otimes \mathrm{End}_0 V)$, *the pair* (V,Φ) *has regular spectral curve.* □

The $\mathbb{C}^*$-action which factors out these 1-dimensional fibres over $\mathfrak{U}$ is actually well-defined on $\bar{\mathcal{X}}_0^{reg}$ by $(V,\Phi) \to (V,c\Phi)$ and has a quasi-projective quotient which we shall denote by $\mathfrak{R} = \bar{\mathcal{X}}_0^{reg}$, which is consequently *birational* to N(n,d). The horizontal map in (10) is equivariant with respect to the $\mathbb{C}^*$-action $(a_1,\ldots,a_n) \to (ca_1, c^2a_2, \ldots, c^na_n)$ and one has the following result:

(16) Theorem: *For each even integer* n *and integer* d(mod n) *there exists an irreducible quasi-projective variety* $\mathfrak{R}$ *and a weighted projective space* Υ *of dimension* $\frac{1}{2}n(n-1)(g-1)$ *such that:*

(i) $\mathfrak{R}$ *is birational to* N(n,d)*;*

(ii) $\exists$ *dominant morphism* $\mathfrak{R} \to \Upsilon$ *with generic fibre*

$\mathrm{Jac}(\tilde{X})/\pm 1$ *if* $n=2$*;* $\mathrm{Jac}(\tilde{X})$ *if* $n>2$*;*

where $\tilde{X}$ *is a (variable) smooth irreducible curve of genus* $\tilde{g} = 1 + \frac{1}{2}n(n+1)(g-1)$. □.

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